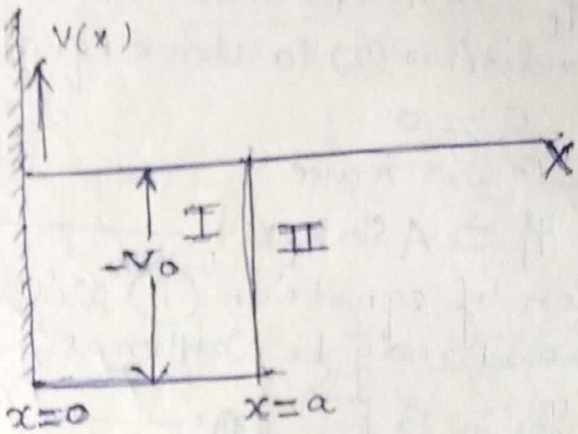
Potential well of finite depth

According to classical Concept, a particle having energy less than zero would be bound inside the potential well, since it would not have the energy to escape. According to quantum mechanics, though there are such bound states, but in general, the possible energies of bound states are not continuous but discrete.

The potential functions are

$$\left.\begin{aligned} V(x) &= -V_0 \quad \text{for } 0<x<a \\ V(x) &= 0 \quad \text{for } x>a \\ V(x) &= \infty \quad \text{for } x<0 \end{aligned}\right\} ; \quad \text{——} (1)$$

The figure is shown as :—



In other words, the potential energy for $x>a$ is everywhere equal to zero and $-V_0$ within the region of width a near the origin of Co-ordinate system, which region we Call the well. For $x<0$ the potential energy is infinite, so that the particle is reflected at $x=0$ and hence the probability that a particle will be at $x=0$ is zero i.e

$$\psi(x) = 0 \text{ at } x=0 ; \quad \text{——} (2)$$

To solve the problem, let us write two Schrodinger's equations one for each region.

The Schrodinger equation for I region is

$$\frac{d^2\psi_1}{dx^2} + \frac{2m}{\hbar^2}(V_0 - E)\psi_1 = 0 \quad \text{——} (3)$$

(Since E and V_0 both are negative)

The Schrodinger equation for II region is

$$\frac{d^2\psi_2}{dx^2} - \frac{2m}{\hbar^2}E\psi_2 = 0 \quad \text{——} (4)$$

Here E is the energy of the particle inside the well and is negative. Substituting

$$\left.\begin{aligned} \frac{2m}{\hbar^2}(V_0-E) &= K_1^2 \\ \text{and} \quad \frac{2mE}{\hbar^2} &= K_2^2 \end{aligned}\right\} \text{———} (5)$$

Equations (3) and (4) become

$$\frac{d^2\psi_1}{dx^2} + K_1^2\psi_1 = 0 \text{———} (6)$$

$$\text{and} \quad \frac{d^2\psi_2}{dx^2} - K_2^2\psi_2 = 0 \; ; \text{———} (7)$$

The general solution of equation (6) is written as

$$\psi_1 = A \sin K_1 x + C \cos K_1 x$$

Applying condition (2) to above equation, we get

$$C = 0$$

So that we have

$$\psi_1 = A \sin K_1 x \; ; \text{———} (8)$$

The solution of equation (7) will be of hyperbolic form and may be written as —

$$\psi_2 = B e^{-K_2 x} \; ; \text{———} (9)$$

Here the exponent of the form $e^{K_2 x}$ is not used because when $x \to \infty$, ψ_2 should be finite or zero. The form $e^{K_2 x}$ gives $\psi = \infty$ as $x \to \infty$ and hence an unacceptable solution.

As ψ and $\frac{d\psi}{dx}$ must be continuous at $x = a$ i.e

$$\left.\begin{aligned} \psi_1 &= \psi_2 \quad \text{at } x = a \text{———} (A) \\ \frac{d\psi_1}{dx} &= \frac{d\psi_2}{dx} \quad \text{at } x = a \text{———} (B) \end{aligned}\right\} \text{———} (10)$$

Diff. equation (8) and (9), we get

$$\frac{d\psi_1}{dx} = A K_1 \cos K_1 x \text{———} (11)$$

$$\frac{d\psi_2}{dx} = -B K_2 e^{-K_2 x} \text{———} (12)$$

Applying boundary condition ~~to~~ (10 A) to equation (8) and (9) and (10 B) to equation

(11) and (12), we get (3)

$A \sin K_1 a = B e^{-K_2 a}$ ——— (13)

$A K_1 \cos K_1 a = -B K_2 e^{-K_2 a}$; ——— (14)

Dividing (14) by (13) we get

$$\frac{\cancel{A} K_1 \cos K_1 a}{\cancel{A} \sin K_1 a} = \frac{-\cancel{B} K_2 \cancel{e^{-K_2 a}}}{\cancel{B}\, \cancel{e^{-K_2 a}}}$$

$$K_1 \cot K_1 a = -K_2$$

$$\cot K_1 a = -\frac{K_2}{K_1}; \quad \text{——— (15)}$$

But $\sin K_1 a = \dfrac{1}{\sqrt{1 + \cot^2 K_1 a}}$

$$\therefore \sin K_1 a = \pm \frac{1}{\sqrt{1 + \left(\frac{K_2}{K_1}\right)^2}}$$

$$\left[\frac{K_2^2}{K_1^2} = \frac{E}{V_0 - E} \right]$$

$$= \pm \frac{1}{\sqrt{1 + \left(\frac{E}{V_0 - E}\right)}} \quad (\text{using } 5)$$

$$\sin K_1 a = \pm \sqrt{\frac{V_0 - E}{V_0}} \quad \text{——— (16)}$$

$$= \pm \sqrt{\frac{T}{V_0}} \quad \text{where } T = V_0 - E \quad \text{(17)}$$

is the kinetic energy of the particle inside the well.

we have

$$\sqrt{\frac{2m(V_0 - E)}{\hbar^2}} = K_1$$

$$\sqrt{V_0 - E} = \sqrt{\left(\frac{\hbar^2}{2m}\right)}\, K_1 = \frac{\hbar}{\sqrt{2m}} K_1 \quad \text{——— (18)}$$

Putting this value in (18)

$$\boxed{\sin K_1 a = \pm \frac{\hbar}{\sqrt{2mV_0}} K_1} \quad \text{——— (19)}$$